KAPPA-CONTRACTION FROM $SU_q(2)$ TO $E_\kappa(2)$*

J. Sobczyk

Institute for Theoretical Physics, Wrocław University, pl. Maxa Borna 9
50-204 Wrocław, Poland; e-mail: jsobczyk@proton.ift.uni.wroc.pl

ABSTRACT

We present contraction prescription of the quantum groups: from $SU_q(2)$ to $E_\kappa(2)$. Our strategy is different then one chosen in ref. [P. Zaugg, J. Phys. A **28** (1995) 2589]. We provide explicite prescription for contraction of a, b, c and d generators of $SL_q(2)$ and arrive at * Hopf algebra $E_\kappa(2)$.

The study of deformations of the $E(2)$ - two-dimensional euclidean group (or its dual $U(e(2))$) reveals many interesting features of the theory of quantum groups. It turns that they admits different deformations according to different Lie bialgebra structures on $e(2)$. One of the most fruitful (and historically the first) approaches to contruct quantum deformations of $U(e(2))$ is that of contraction from $U_q(sl(2))$ or $SL_q(2)$ [1] - [3]. In this way two different deformations of $U(e(2))$ were discovered [1]. It is also known that with the help of contraction it is possible to obtain one of the two dual quantum deformations of $E(2)$ [3]. With the second quantization of $E(2)$, as far as the contraction is concerned,

* Submitted to the Proceedings of the 4-th Colloqium *Quantum Groups and Integrable Systems*, Prague June 1995

the situation is less clear. This contraction involves the renormalization of the deformation parameter and is a $D = 2$ analogue of the κ-deformation of the $D = 4$ Poincaré algebra. One possible approach to the problem presented in ref. [4] takes advantage of the quantum plane and is essentially a contraction of $U_q(O(3))$ quantum group. In this contribution we would like to follow the more straithforward route and present a contraction scheme in terms of a, b, c and d generators of the quantum $SL_q(2)$ together with the $*$ operation making it $SU_q(2)$. They are subject to the standard relations

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad RT_1T_2 = T_2T_1R \tag{1}$$

with

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{2}$$

$$\Delta(T_{ij}) = T_{ik} \otimes T_{kj} \tag{3}$$

$$S(T) = T^{-1}, \qquad T^*_{jk} = S(T_{kj}) \tag{4}$$

We will show that the following prescription defines a required contraction $SU_q(2) \to E_\kappa(2)$ in the limit as the contraction parameter $\rho \to \infty$

$$a = K + \frac{L}{\rho} + ... \qquad b = M + \frac{iN}{\rho} + ... \tag{5}$$

$$c = M - \frac{iN}{\rho} + ... \qquad q = e^{\frac{1}{\kappa\rho}} \tag{6}$$

These formulas were suggested (but in rather indirect way) by the duality relations between $Sl_q(2)$ and $U_q(sl(2))$ and the contraction prescription $U_q(sl(2)) \to U_\kappa(e(2))$. We did not write down the contraction behaviour of d as it is dictated by the determinant relation

$$ad - qbc = 1 \tag{7}$$

One can show:

$$d = K^{-1}(1 + M^2) + \frac{1}{\rho}\left(\frac{1}{\kappa}K^{-1}M^2 - K^{-1}LK^{-1}(1 + M^2)\right) + ... \tag{8}$$

In order to find this expression one has to assume that operator a (and then also K) is invertible. This is a typical situation in contraction of quantum groups. Some discussion

related to this point can be found in [2], [5]. We start now to analyze relations imposed on K, L, M and N by (1) and (3). In the zero order in $\frac{1}{\rho}$ (3) give rise to

$$K^2 = 1 + M^2 \tag{9}$$

Next terms linear in $\frac{1}{\rho}$ impose the condition:

$$[L, K] = \frac{1}{\kappa} M^2 \tag{10}$$

Taking all that into account one can write coproducts for our set of variables

$$\Delta(K) = K \otimes K + M \otimes M \tag{11}$$

$$\Delta(M) = K \otimes M + M \otimes K \tag{12}$$

$$\Delta(L) = K \otimes L + L \otimes K + iN \otimes M - iM \otimes N \tag{13}$$

$$\Delta(N) = K \otimes N - iL \otimes M + iM \otimes L + N \otimes K \tag{14}$$

The star operation inherited from $SL_q(2)$ acts on them as follows:

$$K^* = K, \qquad L^* = -L, \qquad M^* = -M, \qquad N^* = -N - \frac{iM}{\kappa} \tag{15}$$

It is useful to introduce linear combinations of K, L, M and N, namely $K \pm M$ and $L \mp \frac{M}{2\kappa} \pm iN$. Their coproducts are:

$$\Delta(K \pm M) = (K \pm M) \otimes (K \pm M) \tag{16}$$

$$\Delta(L \mp \frac{M}{2\kappa} \pm iN) = (L \mp \frac{M}{2\kappa} \pm iN) \otimes (K \pm M) + (K \mp M) \otimes (L \mp \frac{M}{2\kappa} \pm iN) \tag{17}$$

Now we investigate what follows from the contraction of quadratic relations given in (1). $ab = qba$, again up to terms $\frac{1}{\rho}$, implies

$$[K, M] = 0 \tag{18}$$

$$[K, iN] + [L, M] = \frac{1}{\kappa} MK \tag{19}$$

Next, $ac = qca$ implies

$$[L, M] - [K, iN] = \frac{1}{\kappa} MK \tag{20}$$

Altogether we obtain

$$[K, N] = 0 \tag{21}$$

$$[L, M] = \frac{1}{\kappa} MK \tag{22}$$

Finally, $bc = cb$ is transformed into

$$[M, N] = 0 \tag{23}$$

In the variables $K \pm M$ and $L \mp \frac{M}{2\kappa} \pm iN$ we first observe that

$$(K + M)(K - M) = (K - M)(K + M) = 1 \tag{24}$$

Then we calculate

$$[L \mp \frac{M}{2\kappa} \pm iN, K + M] = \frac{1}{2\kappa} \left((K + M)^2 - 1\right) \tag{25}$$

$$[L \mp \frac{M}{2\kappa} \pm iN, K - M] = \frac{1}{2\kappa} \left((K - M)^2 - 1\right) \tag{26}$$

On the other hand we are yet unable to calculate the commutator

$$[L - \frac{M}{2\kappa} + iN, L + \frac{M}{2\kappa} - iN] \tag{27}$$

as we do not know what is $[L, N]$. From the form of (5)-(6) it is clear that we really need the analysis of terms of order $\frac{1}{\rho^2}$. We shall come back to this point in the discussion. Let us introduce still another set of variables

$$\eta = (L - \frac{M}{2\kappa} + iN)(K - M) \tag{28}$$

$$\bar{\eta} = -(K + M)(L + \frac{M}{2\kappa} - iN) \tag{29}$$

They are defined in such a way that $\eta^* = \bar{\eta}$. On the other hand $(K + M)^* = K - M$. Let us call $(K + M)^2 = e^{i\alpha}$. We derive

$$\Delta(\eta) = \eta \otimes 1 + e^{-i\alpha} \otimes \eta \tag{30}$$

$$\Delta(\bar{\eta}) = \bar{\eta} \otimes 1 + e^{i\alpha} \otimes \bar{\eta} \tag{31}$$

$$\Delta(e^{i\alpha}) = e^{i\alpha} \otimes e^{i\alpha} \tag{32}$$

$$[\eta, e^{i\alpha}] = \frac{1}{\kappa}(e^{i\alpha} - 1) \tag{33}$$

$$[\bar{\eta}, e^{i\alpha}] = \frac{1}{\kappa}(e^{i\alpha} - e^{2i\alpha}) \tag{34}$$

As we have said before, for completeness we still need the commutator $[\eta, \bar{\eta}]$. If we look at the coproducts of η and $\bar{\eta}$ one discovers that the consistent choice is

$$[\eta, \bar{\eta}] = \frac{1}{\kappa}(\bar{\eta} + \eta) \tag{35}$$

It is interesting that a similar problem of a necessity of completing the structure of the quantum group appeared before in the paper [6] where the authors tried to determine the quantum releations of $E_q(2)$ by means of the numerical R matrix satisfying Yang Baxter equation. In our case the remaining relation can certainly be obtained by analysis of the terms quadratic in the contraction parameter. This analysis is however very tedious and not particulary iluminating. Let us still mention that expressions for the antipode and counit can be also obtained along the lines explained above. In this way we reproduced the relations defining the quantum group $E_\kappa(2)$. Expressions (30)- (35) can be compared to those obtained before with a help of other techniques e.g in refs. [6] [7].

References

[1] E. Celeghini, R. Giachetti, R. Sorace and M. Tarlini, J. Math. Phys. **31** (1990) 2548; *Contractions of quantum groups* in *Quantum Groups*, Lecture Notes in Mathematics 1510, 221, Springer Verlag 1992

[2] S. Woronowicz, Lett. Math. Phys. **23** (1991) 251

[3] P. Schupp, P. Watts and B. Zumino, Lett. Math. Phys. **24** (1992) 637

[4] P. Zaugg, J. Math. Phys. **36** (1995) 1547; J. Phys. A **28** (1995) 2589

[5] D. Ellinas and J. Sobczyk, J. Math. Phys. **36** (1995) 1404

[6] A. Ballesteros, E. Celeghini, R. Giachetti, R. Sorace and M. Tarlini, J. Phys. A **26** (1993) 7495

[7] P. Maślanka, J. Math. Phys. **35** (1994) 1976